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Some formulas for the coefficients of Drinfeld modular forms ☆

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Abstract

We obtain some formulas for t-expansion coefficients of meromorphic Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$. Let $j(z)$ be the Drinfeld modular invariant. As an application we show that the values of $j(z)$ at points in the divisor of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$ are algebraic over $\mathbb{F}_q(T)$.

MSC: 11F52

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1. Introduction

Let $A = \mathbb{F}_q[T]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ and $K = \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((1/T))$ be the completion of K at $1/T$ and C the completion of the algebraic closure of K_∞. Let $\Omega = C - K_\infty$ be the Drinfeld upper half plane. The Drinfeld modular invariant $j(z)$ has many interesting arithmetic properties. For example, for arguments $\tau \in \Omega$ that are imaginary quadratic over K, the values $j(\tau)$ are algebraic

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integers over A. Moreover, if $a\tau^2+b\tau+c=0$ and b^2-4ac is a field discriminant, then $j(\tau)$ generates the Hilbert class field of $K(\tau)$ (see [3] or [4]). Such integers are called singular invariants. Dorman [3] studied the prime factorization of such invariants. Here we consider the values of a specific sequence of Drinfeld modular functions $j_n(z)$ for $\Gamma = GL_2(A)$.

Recently in [2], Bruinier et al. studied the values of elliptic modular functions J_n where $J_1 = J - 744$ and J is the usual elliptic modular function for $SL_2(\mathbb{Z})$. It is natural to investigate the analogue in the function field setting. We consider sums of values of $j_n(z)$ over divisors of Drinfeld meromorphic modular forms for Γ. Theorem 3.5 provides a very useful link relating the values of $j(z)$ to the arithmetic of t-expansion coefficients of Drinfeld modular forms for Γ. It also gives an explicit formula for the action of the operator $\vartheta = -t^2 d/dt$ on any Drinfeld modular forms for $GL_2(A)$. This operator ϑ is analogous to the Ramanujan's theta operator. By using the residue theorem we find some formulas for t-expansion coefficients of any Drinfeld modular forms for Γ (Theorem 3.2). The idea to use the residue theorem comes from the paper [1]. As an application we show that the values of $j(z)$ at points in the divisor of Drinfeld modular forms for Γ are algebraic over K (Corollary 3.6).

2. Preliminaries

Let $L = \tilde{\pi}A$ be the rank 1 A-lattice in C associated to the Carlitz module ρ. We let $e_A(z)$ be the exponential function associated to A, i.e.,

$$e_A(z) := z \prod_{\lambda \in A-\{0\}} \left(1 - \frac{z}{\lambda}\right)$$

and $t = t(z) := 1/(\tilde{\pi} e_A(z))$, $s = s(z) := t(z)^{q-1}$. For any nonzero $a \in A$ we define $t_a = t_a(z) := t(az)$. A meromorphic Drinfeld modular form for Γ of weight k and type l (where $k \geqslant 0$ is an integer and l is a class in $\mathbb{Z}/(q-1)$) is a meromorphic function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k f(z)$ for any $\gamma \in \Gamma$,
(ii) f is meromorphic at the cusp ∞.

If f is a meromorphic Drinfeld modular form of weight k and type l, then t-expansion of f is of the form

$$f = \sum_i a_f((q-1)i + l)t^{(q-1)i+l}.$$

Here and in what follows, we chose the representative l in the class with $0 \leqslant l < q-1$. Indeed, let ε be a primitive $(q-1)$th root of unity in $\mathbb{F}_q$. If $f(z) = \Sigma_n a_f(n)t^n$, then $f(\varepsilon z) = \varepsilon^{-l} f(z)$. This implies that $\varepsilon^{l-n} = 1$ because $t(\varepsilon z) = \varepsilon^{-1} t(z)$ for each n.

Hence $n \equiv l \bmod (q-1)$ for each n. When $k = l = 0$ we call it a Drinfeld modular function for Γ. Let M_k^l be the C-vector space of meromorphic modular forms for Γ of weight k and type l.

Let $A_+ = \{a \in A : a \text{ is monic}\}$. Let $E = E(z) := \sum_{a \in A_+} a t_a(z)$. Then E is a conditionally convergent two-dimensional lattice sum

$$\frac{1}{\tilde{\pi}} \sum_{a \in A_+} \left(\sum_{b \in A} \frac{a}{az+b} \right)$$

and it may be considered as an analogue of the "false Eisenstein series of weight 2" in the classical theory.

We define $\vartheta = \tilde{\pi}^{-1} d/dz$ and $\partial_k = \vartheta + k \cdot E$ as operators on M_k^l (see [5]). A direct computation shows that if $f \in M_k^l$, then $\partial_k f = \vartheta(f) + k \cdot E \cdot f \in M_{k+2}^{l+1}$ and $\vartheta(f)/f + k \cdot E \in M_2^1$. We further observe that $\vartheta(\sum_{n=h}^{\infty} b(n) t^n) = \sum_{n=h}^{\infty} -nb(n) t^{n+1}$.

3. Drinfeld modular forms and the action of the operator ϑ

For any $z \in \Omega$, we let $\Lambda_z = Az + A$, a rank 2 A-lattice in C. It induces a Drinfeld module ϕ^z of rank 2 determined by

$$\phi_T^z(X) = TX + g(z) X^q + \Delta(z) X^{q^2}.$$

The j-invariant $j(z)$ of ϕ^z is defined to be $g(z)^{q+1}/\Delta(z)$, which is a Drinfeld modular function for Γ. The Drinfeld modular functions for Γ which are holomorphic on Ω are exactly the polynomials in $j(z)$. Since $j(z) = -1/s + \sum_{n=0}^{\infty} c(n) s^n$, for each positive integer m, there exists a unique Drinfeld modular function $j_m(z)$ which has the s-expansion as follows:

$$j_m(z) = \frac{1}{s^m} + \sum_{n=1}^{\infty} c_m(n) s^n.$$

Indeed, $j_m(z)$ is a polynomial in $j(z)$ of degree m with coefficients in A and its leading coefficient is $(-1)^m$. When $q = 2$, the first few $j_m(z)$ are

$$\begin{aligned}
j_1(z) &= j(z) + 1 + T + T^2, \\
j_2(z) &= j^2(z) + 1 + T^2 + T^4, \\
j_3(z) &= j^3(z) + (1 + T^2 + T^4) j^2(z) + j(z) + T^8 + T + 1, \\
j_4(z) &= j^4(z) + 1 + T^4 + T^8.
\end{aligned}$$

For any $G(z) \in M_2^1$, $\omega := G(z)dz$ is a 1-form on the compactification $\overline{\Gamma\backslash\Omega}$ of $\Gamma\backslash\Omega$. Let $G(z) = \sum_{n=n_0}^{\infty} a(n)t^n$ be the t-expansion of $G(z)$ and $\overline{\Gamma\backslash\Omega} - \Gamma\backslash\Omega = \{\infty\}$. Let $\pi : \Omega \to \Gamma\backslash\Omega$ be the quotient map. Then we have

Lemma 3.1. (i) $\mathrm{Res}_\infty\, \omega = -a(1)/\tilde{\pi}$.
(ii) $\mathrm{Res}_\tau\, G(z) = \mathrm{Res}_{\pi(\tau)}\, \omega$ *for each* $\tau \in \Omega$.

Proof. (i) follows from the simple fact that $-\tilde{\pi}t^2\, dz = dt$. For any ordinary point $\tau \in \Omega$, (ii) is obvious. Suppose $\tau \in \Omega$ is an elliptic point. Let Γ_τ be the stabilizer of τ in Γ and $Z(K)$ be the center of scalar matrices. Let $e_\tau = |\Gamma_\tau/(\Gamma_\tau \cap Z(K))|$. Indeed, $e_\tau = q+1$ because τ is an elliptic point. We choose uniformizers x and y on Ω and $\Gamma\backslash\Omega$, respectively, with $x^{e_\tau} = y$. Then $dy = e_\tau x^{e_\tau - 1}dx = x^{e_\tau - 1}dx$, which gives the assertion (ii). $\square$

For each integer $n \geqslant 1$ we define the Y_n's by the following recursion formula:

$$Y_1 = -X_1, \quad Y_n + X_1Y_{n-1} + X_2Y_{n-2} + \cdots + X_{n-1}Y_1 + n \cdot X_n = 0\, (n \geqslant 2).$$

Here X_i is an indeterminate for each positive integer i. Then $Y_n + n\cdot X_n$ is a polynomial in $X_1, X_2, \ldots, X_{n-1}$ with integer coefficients. Let $F_{n-1}(X_1, \ldots, X_{n-1}) := Y_n + n \cdot X_n (\mathrm{mod}\, p) \in \mathbb{F}_p[X_1, \ldots, X_{n-1}]$, where $\mathbb{F}_p$ is the prime field of $\mathbb{F}_q$. The first few polynomials $F_{n-1}(X_1, \ldots, X_{n-1})$ are

$$F_1(X_1) = X_1^2,$$
$$F_2(X_1, X_2) = -X_1^3 + 3X_1X_2,$$
$$F_3(X_1, X_2, X_3) = X_1^4 - 4X_1^2X_2 - 2X_1X_3 + 2X_2^2,$$
$$F_4(X_1, X_2, X_3, X_4) = -X_1^5 + 3X_1^3X_2 + X_1^2X_3 - 5X_1X_2^2 + 5X_2X_3 + 5X_4X_1.$$

Theorem 3.2. *Let f be any meromorphic Drinfeld modular form of weight k and type l for Γ with the t-expansion*

$$f(z) = t^{(q-1)h+l} + \sum_{n=h+1}^{\infty} a_f((q-1)n+l)t^{(q-1)n+l}.$$

Then for each integer $n > 1$, we have

$$\begin{aligned} &n \cdot a_f((q-1)(n+h)+l) \\ &\quad = F_{n-1}(a_f((q-1)(h+1)+l), \ldots, a_f((q-1)(n+h-1)+l)) \\ &\qquad -k\sum_{a\in A_+} a \cdot \{t_a\}_{(q-1)n+1} + \sum_{\pi(\tau)\in\Gamma\backslash\Omega} \mathrm{ord}_\tau\, f \cdot j_n(\tau), \end{aligned}$$

where $\{t_a\}_{(q-1)n+1}$ is the coefficient of $t^{(q-1)n+1}$ in t_a.

Proof. For a positive integer m, let $G_m(z) = (\vartheta(f)/f + kE)j_m(z)$ and $\omega_m = G_m(z)dz$. Then ω_m is a 1-form on $\overline{\Gamma\backslash\Omega}$. We calculate the residue of ω_m at each point of $\overline{\Gamma\backslash\Omega}$. At first, we consider the cusp ∞. Since

$$\frac{\vartheta(f)}{f} = (h-l)t - \sum_{n=1}^{\infty} b_{(q-1)n} t^{(q-1)n+1} \tag{3.1}$$

for some $b_{(q-1)n} \in C$, we have

$$\begin{aligned} G_m(z) &= \left(\frac{\vartheta(f)}{f} + kE\right) j_m(z) \\ &= \left((h-l)t - \sum_{n=1}^{\infty} b_{(q-1)n} t^{(q-1)n+1} + k \sum_{a\in A_+} a t_a\right)\left(\frac{1}{s^m} + \sum_{n=1}^{\infty} c_m(n) s^n\right) \\ &= \cdots + \left(-b_{(q-1)m} + k \sum_{a\in A_+} a\cdot \{t_a\}_{(q-1)m+1}\right) t + \cdots. \end{aligned}$$

Thus by Lemma 3.1 (i), we obtain

$$\operatorname{Res}_\infty \omega_m = \frac{1}{\tilde{\pi}}\left(b_{(q-1)m} - k\sum_{a\in A_+} a\cdot\{t_a\}_{(q-1)m+1}\right).$$

Let $\tau \in \Omega$. Since $E(z)$ and $j_m(z)$ are holomorphic on Ω, we have

$$\begin{aligned} \operatorname{Res}_\tau G_m(z) &= \operatorname{Res}_\tau \left(\frac{\vartheta(f)}{f} + kE\right) j_m(z) \\ &= \operatorname{Res}_\tau \frac{\vartheta(f)}{f} j_m(z) = \frac{\operatorname{ord}_\tau f \cdot j_m(\tau)}{\tilde{\pi}}, \end{aligned}$$

where $\operatorname{ord}_\tau f$ means the order of f in the prime field $\mathbb{F}_p$ of $\mathbb{F}_q$. Hence by Lemma 3.1 (ii), we obtain

$$\operatorname{Res}_{\pi(\tau)} \omega_m = \frac{\operatorname{ord}_\tau f \cdot j_m(\tau)}{\tilde{\pi}}.$$

Consequently the residue theorem ($\sum_{\mu\in\overline{\Gamma\backslash\Omega}} \operatorname{Res}_\mu \omega_m = 0$) shows

$$b_{(q-1)m} = k\sum_{a\in A_+} a\cdot\{t_a\}_{(q-1)m+1} - \sum_{\pi(\tau)\in\Gamma\backslash\Omega} \operatorname{ord}_\tau f\cdot j_m(\tau). \tag{3.2}$$

On the other hand, from (3.1), we have that $b_{q-1} = -a_f((q-1)(h+1)+l)$ and

$$\begin{aligned}
&b_{(q-1)n} + b_{(q-1)(n-1)} \cdot a_f((q-1)(h+1)+l) + \cdots \\
&\quad + b_{q-1} \cdot a_f((q-1)(n+h-1)+l) + a_f((q-1)(n+h)+l) \cdot n = 0.
\end{aligned}$$

Thus for any integer $n \geqslant 2$,

$$\begin{aligned}
b_{(q-1)n} = {} & F_{n-1}\big(a_f((q-1)(h+1)+l), \ldots, a_f((q-1)(n+h-1)+l)\big) \\
& - n \cdot a_f((q-1)(n+h)+l). \qquad (3.3)
\end{aligned}$$

By combining (3.2) with (3.3), we get the assertion. $\square$

Corollary 3.3. *We have*

$$a_f((q-1)(h+1)+l) = -ks_q + \sum_{\pi(\tau)\in\Gamma\backslash\Omega} \mathrm{ord}_\tau f \cdot j_1(\tau),$$

where s_q *is the sum over the elements of* $\mathbb{F}_q$, *and is* 0 *except for* $q = 2$, *where it is* 1.

Proof. It follows from

$$\begin{aligned}
a_f((q-1)(h+1)+l) = -b_{q-1} &= -k \sum_{a\in A_+} a \cdot \{t_a\}_q + \sum_{\pi(\tau)\in\Gamma\backslash\Omega} \mathrm{ord}_\tau f \cdot j_1(\tau) \\
&= -ks_q + \sum_{\pi(\tau)\in\Gamma\backslash\Omega} \mathrm{ord}_\tau f \cdot j_1(\tau). \qquad \square
\end{aligned}$$

Example 3.4. Let $\Delta(z)$ be the Drinfeld discriminant function having the t-expansion as follows:

$$-\tilde{\pi}^{1-q^2}\Delta(z) = \sum_{m=1}^{\infty} a((q-1)m)t^{(q-1)m}.$$

Since $\Delta(z)$ has no zeros and no poles on Ω and $a(q-1) = 1$, by Theorem 3.2, we have

$$\begin{aligned}
m \cdot a((q-1)(m+1)) = {} & F_{m-1}(a((q-1)2), \ldots, a((q-1)m)) \\
& - k \sum_{a\in A_+} a \cdot \{t_a\}_{(q-1)m+1}.
\end{aligned}$$

Let τ be a fixed point of Ω. Let $H_\tau(z) := \vartheta(j(z))/(j(z) - j(\tau)) \in M_2^1$. By the proof of Theorem 3.2 (Eq. (3.2), and letting $f(z) = j(z) - j(\tau)$), we see that $H_\tau(z)$ has the t-expansion as follows:

$$H_\tau(z) = -t + \sum_{n=1}^{\infty} j_n(z) t^{(q-1)n+1}.$$

For any $f \in M_k^l$, we define $J_f := \vartheta(f)/f + kE \in M_2^1$.

Theorem 3.5. *Let $f \in M_k^l$ be given as in Theorem* 3.2. *Then J_f has the t-expansion as follows*:

$$J_f = (k + h - l)t + \sum_{\pi(\tau) \in \Gamma \backslash \Omega} (\mathrm{ord}_\tau f)(H_\tau(z) + t).$$

Proof. We use the same notations as in the proof of Theorem 3.2. From (3.1) and (3.2), we have

$$\begin{aligned}
J_f &= (h - l)t - \sum_{n=1}^{\infty} b_{(q-1)n} t^{(q-1)n+1} + kE \\
&= (h - l)t - k \sum_{n=1}^{\infty} \sum_{a \in A_+} a \cdot \{t_a\}_{(q-1)n+1} \cdot t^{(q-1)n+1} \\
&\quad + \sum_{n=1}^{\infty} \sum_{\pi(\tau) \in \Gamma \backslash \Omega} \mathrm{ord}_\tau f \cdot j_n(\tau) \cdot t^{(q-1)n+1} + kE \\
&= (h - l)t + k \sum_{a \in A_+} a \cdot \{t_a\}_1 - kE + \sum_{\pi(\tau) \in \Gamma \backslash \Omega} (\mathrm{ord}_\tau f)(H_\tau(z) + t) + kE \\
&= (k + h - l)t + \sum_{\pi_\tau \in \Gamma \backslash \Omega} (\mathrm{ord}_\tau f)(H_\tau(z) + t). \qquad \square
\end{aligned}$$

Theorem 3.5 easily reveals some algebraic information about the $j_n(z)$ evaluated at the finite points of the divisor of any meromorphic Drinfeld modular form.

Corollary 3.6. *Let $\tau \in \Omega$ be a point for which* $\mathrm{ord}_\tau(f) \neq 0$. *Suppose that the t-expansion coefficients of f are algebraic over $\mathbb{F}_q(T)$. Then $j(\tau)$ is algebraic over $\mathbb{F}_q(T)$.*

Finally, we express H_τ in terms of the well-known Drinfeld modular forms. Let $h(z)$ be the Poincaré series $P_{q+1,1}(z)$ (see [5, p. 681]) and $g_{\mathrm{new}}(z)$ be the Drinfeld modular form $\tilde{\pi}^{1-q}(T^q - T)E^{(q-1)}(z)$, where $E^{(q-1)}(z)$ is the Eisenstein series of weight $q-1$.

Proposition 3.7. *For any* $\tau \in \Omega$, *we have*

$$H_\tau(z) = \frac{g_{\mathrm{new}}(z)^q}{(j(\tau) - j(z))h(z)^{q-2}}.$$

Especially if τ *is an elliptic point*, *then*

$$H_\tau(z) = \frac{h(z)}{g_{\mathrm{new}}(z)}.$$

Proof. Let $\Delta_{\mathrm{new}}(z) = -t^{q-1} + \cdots$ be the normalized Drinfeld discriminant function. Since $\partial_{q^2-1}\Delta_{\mathrm{new}}(z) = 0$ and $\partial_{q-1} g_{\mathrm{new}}(z) = h(z)$ (see [5, p. 687 and 688]),

$$\begin{aligned} g_{\mathrm{new}}(z)^{q+1}\frac{\vartheta(j(z))}{j(z)} &= \Delta_{\mathrm{new}}(z)\vartheta(j(z)) + j(z)\partial_{q^2-1}\Delta_{\mathrm{new}}(z) \\ &= \partial_{q^2-1}(\Delta_{\mathrm{new}}(z)j(z)) \\ &= g_{\mathrm{new}}(z)^q h(z) \end{aligned}$$

which implies $\vartheta(j(z))/j(z) = h(z)/g_{\mathrm{new}}(z)$. By using the fact that $j(z)h(z)^{q-1} = -g_{\mathrm{new}}(z)^{q+1}$ [5, p. 688], we obtain

$$H_\tau(z) = \frac{\vartheta(j(z))}{j(z) - j(\tau)} = \frac{g_{\mathrm{new}}(z)^q}{(j(\tau) - j(z))h(z)^{q-2}}.$$

Especially if τ is an elliptic point, we have

$$H_\tau(z) = \frac{h(z)}{g_{\mathrm{new}}(z)}. \qquad \square$$

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